Image Smoothing and Segmentation by Graph Regularization

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Abstract. We propose a discrete regularization framework on weighted graphs of arbitrary topology, which leads to a family of nonlinear filters, such as the bilateral filter or the TV digital filter. This framework, which minimizes a loss function plus a regularization term, is parameterized by a weight function defined as a similarity measure. It is applicable to several problems in image processing, data analysis and classification. We apply this framework to the image smoothing and segmentation problems.

1 Introduction

Image smoothing, denoising and segmentation are fundamental problems of computer vision. The goal of image smoothing and denoising is to remove spurious details and/or noise for a given possibly corrupted image, while maintaining essential features such as edges. The goal of segmentation is to divide a given image into parts that belong to distinct objects in the image. There exists several methods to solve these problems. The variational ones, based on regularization, are particularly well suited to impose constraints on the solution, such as regularity. These methods, solved with partial differential equations (PDE), constitute a significant framework in image processing and data analysis. In the case of an image regularization, a classical methodology first supposes the image to be defined on a continuous domain. Then it considers a continuous variational function which typically involves a regularization term (internal energy), and a constraint term (external energy). The problem is formalized by a minimization problem which can be solved by finding the steady-state solution of a heat equation corresponding to the Euler-Lagrange equation. Finally, the resulting PDE are numerically discretized [1], [2].

However, many data can be represented by graphs of arbitrary or complex topology. With these representations, the continuous regularization cannot work. The idea is to consider a discrete regularization on graphs, which can be reduced to solve linear systems or nonlinear systems by iterative methods. This was proposed for images represented by grid graphs [3]. For example, the total variation (TV) digital filter [4], which is a discretization of the continuous one, is used for denoising and enhancing images, or more generally data living on graphs. In the context of data classification, a discrete regularization method is applied on weighted graphs, using discrete differential operators [5], [6], [7]. We propose a regularization framework on weighted graphs of arbitrary topology, which corresponds to a family of nonlinear filters. This family includes the bilateral filter [8] and the TV digital filter. It is parameterized by a weight function defined as a similarity measure. Each filter can be implemented by a simple and fast algorithm. We apply our framework on the image smoothing, denoising and segmentation problems.

This article is organized as follows: In Section 2, we present differential geometry on weighted graphs, which is similar to the one introduced in [6]. In Section 3, we present discrete regularizations on graphs. In Section 4, we present algorithms for image filtering and segmentation by the construction of a graph corresponding to an initial adapted partition and by a diffusion on this graph. The partition of the graph is realized by an energy partition [9].

2 Differential Geometry on Weighted Graphs

A *graph* $G = (V, E)$ consists of a finite set V of vertices and a finite set $E \subseteq V \times V$ of edges. We assume G to be undirected, connected, with no self-loops and no multiple edges. Let (u, v) be the edge that connects the vertices u and v, G is *weighted* if it is associated with a weight function $w : E \to \mathbb{R}_+$ satisfying $w(u, v) = w(v, u)$, for all the edges in E. The *degree* function $d_w : V \to \mathbb{R}_+$ of a vertex v, is a measure on the neighborhood of v: $d_w(v) = \sum_{u \sim v} w(u, v)$, where $u \sim v$ denotes all the vertices u connected to v by an edge of E.

Let $H(V)$ be the Hilbert space of real-valued functions $f : V \to \mathbb{R}$. Similarly define $H(E)$, the *graph gradient* operator $\bigtriangledown : H(V) \to H(E)$ of f on an edge (u, v) is:

$$(\bigtriangledown f)(u, v) = \sqrt{\frac{w(u, v)}{d_w(u)}}(f(u) - f(v)). \tag{1}$$

The amplitude of the gradient, or the *local variation* of f at the vertex v, is defined to be:

$$\| \bigtriangledown_v f\|_{H(V)} = \sqrt{\sum_{u \sim v} (\bigtriangledown f)^2(u, v)}. \tag{2}$$

It can be viewed as a measure of the regularity of a function around a vertex. Meanwhile the *global variation* of f (or the 2-Dirichlet form), defined by:

$$R_p(f) = \frac{1}{p} \sum_{v \in V} \| \bigtriangledown_v f\|^p_{H(V)}, \tag{3}$$

measures of the regularity of f over the graph.

The *graph Laplace* operator $\Delta : H(V) \to H(V)$, of f at a vertex v, is defined to be:

$$(\Delta f)(v) = \left.\frac{\partial R_2(f)}{\partial f}\right|_v = d_w(v)f(v) - \sum_{u \sim v} w(u,v)f(u). \tag{4}$$

3 Regularization of Weighted Graphs

Given a graph $G = (V, E)$ and a function $g \in H(V)$, the regularization of G consists in the search of a function $f \in H(V)$, which is not only smooth enough on G, but also close to g. It is an optimization problem formalized by the minimization of a weighted sum of two energy terms:

$$f^* = \arg \min_{f \in H(V)} \left\{ R_p(f) + \frac{\mu}{2} \|f - g\|^2_{H(V)} \right\}, \tag{5}$$

where $R_p(f)$ represents the regularization term defined by (3), and the second term represents the closeness to the function g. The positive constant μ corresponds to the Lagrange relaxation parameter. Since the energy terms in (5) are strictly convex functions, then the optimization problem has a unique solution f^* which satisfies the equation:

$$\frac{\partial R_p(f^*)}{\partial f^*} + \mu(f^* - g) = 0. \tag{6}$$

Depending on the choice of $p \in \mathbb{N}_*$, the equation (6) leads to different kinds of regularizations. In the particular case of $p = 2$, the equation (6) can be considered as the discrete analogue of the Euler-Lagrange equation on a graph. Using the Laplace operator of the equation (4), we rewrite the equation (6) for each vertex of V:

$$(\mu + d_w(v))f^*(v) - \sum_{u \sim v} w(u,v)f^*(u) = \mu g(v), \forall v \in V. \tag{7}$$

This is a system of linear equations in f^* which is strictly positive definite. Its solution is unique and depends on g and μ. Among the existing methods to solve the system (7), the local iterative ones converge to the solution with efficiency, even if the graph has a large size or a complex topology. The Gauss-Jacobi method is the simplest of them. Let n be the iteration step, $f^{(n)}$ be the function f^* at the step n, and $f^{(0)} = g$. At each vertex v of V, the computation of $f^{(n+1)}(v)$ only depends on $f^{(0)}$ and on the values of $f^{(n)}$ in the neighborhood of v. The following equation expresses an iteration of the algorithm:

$$f^{(n+1)}(v) = \frac{1}{\mu + d_w(v)} \sum_{u \sim v} w(u,v) f^{(n)}(u) + \frac{\mu}{\mu + d_w(v)} f^{(0)}(v). \tag{8}$$

The above method is a forced low-pass digital filter. We call it, the *anisotropic weighted Laplace filter* and note it $AWL(n, G, g, \mu)$.

The cases where $p \neq 2$ are not the purpose of this article since they do not use the Laplace operator. As in the case of $p = 2$, they have been used in numerous applications with other definitions of the gradient operator (1), see [6], [7] for example.

4 Application to Image Filtering and Segmentation

4.1 Graph Representation and Energy Partition

Let $g : \mathbb{Z}^2 \rightarrow \mathbb{R}$ be a grey level image of pixels. We modelize g by a weighted graph $G = (V, E)$ such that each vertex of V corresponds to a pixel of g, and the weight function

$$w(v_i, v_j) = exp(-\lambda|g(v_i) - g(v_j)|), \lambda \in \mathbb{R}_+ \tag{9}$$

estimates the similarity between two pixels. Since the proposed framework enables to deal with arbitrary graphs, we experiment two graph representations: (i) regular grid graphs generated by 4-adjacency, and (ii) graphs of arbitrary topology generated by a pre-segmentation of g. In the case (ii), we first modelize g by a regular grid graph generated by 4-adjacency. Then, we compute an energy partition of G which is analogue to the energy partition of the image domain [9]. In the following, we present the mathematical framework associated to the energy partition of graphs.

A *path* $c(u, v)$ is a sequence of vertices $(v_1, \ldots, v_m)$ such that $u = v_1$, $v = v_m$, and $(v_i, v_{i+1}) \in E$ for all $1 \leq i < m$. Let $C_G(u, v)$ be the set of paths connecting u and v. We define the pseudo-metric $\delta : V \times V \rightarrow \mathbb{R}_+$ to be:

$$\delta(u, v) = \min_{c \in C_G(u,v)} \left(\sum_{i=1}^{m-1} w(v_i, v_{i+1}) \right) . \tag{10}$$

Given a finite set of source vertices $S = \{s_1, \ldots, s_k\} \subset V$, the *energy* induced by δ is given by the minimal individual energy: $\delta_S(v) = \inf_{s_i \in S} \delta(s_i, v), \forall v \in V$. Based on the pseudo-metric δ, the *influence zone* of a source vertex s_i is defined to be the set of vertices that are closer to s_i than to any other source vertex: $Z_\delta(s_i, S) = \{v \in S | \delta(s_i, v) \leq \delta(s_j, v), \forall s_j \in S\}$. The *energy partition* of G, with respect to the set of sources S and the pseudo-metric δ, corresponds to the set of influence zones: $E_\delta(S, \Gamma) = \{Z_\delta(s_i, S), \forall s_i \in S\}$.

With these definitions, the image pre-segmentation consists in finding a set of source vertices and a pseudo-metric. We use the set of extrema of the intensity of g as a set of source vertices. To obtain exactly an energy partition which considers the total variation of g along a path, we use the following weight function in (10): $w(u, v) = |g(u) - g(v)|$. Then, the energy partition of the graph represents an approximation of the image, by assigning a model to each influence zone of the partition. The model is determined by the distribution of the graph values on the influence zone. Among the different models, the simplest are the constant ones, as mean or median value of the influence zone. The resultant graph $G' = (V', E')$, is a connectivity graph where $V' = S$ and E' is the set of edges connecting two vertices $s_i, s_j \in S$ if $Z_\delta(s_i, S) \cap Z_\delta(s_j, S) \neq \emptyset$.

4.2 Image Smoothing and Denoising

Given an image g as defined in Section 4.1, an integer n, and two reals λ (for the weight function (9)) and μ, the image g is transformed into an image $f^* =$

$AWF(n, g, G, \mu)$. The method described in Section 3 gives the iterative algorithm of the AWL filter. The action of the filter is illustrated in Fig.1 on a grid graph for denoising, and on an arbitrary graph in Fig.2 for smoothing. The arbitrary graph is a connectivity graph obtained by an energy partition of g.

4.3 Image Segmentation

Given an image g as defined in Section 4.1, an integer n, and three reals λ (for the weight function (9), μ and t, the segmentation algorithm is organized in four main steps:

(i) Pre-segmentation of g from its associated grid graph G, which gives a graph G' and a pre-segmented image g' (see Section 4.1).
(ii) Regularization of G' by the iterative algorithm: $f^* = AWL(n, G', g', 0)$.
(iii) We cut the edges of G' which have a weight less than a fixed threshold t (the weight is computed from f^*).
(iv) We merge the influence zones of g' that remain connected by an edge.

The segmentation algorithm is illustrated in Fig.3, where g' is a model of g based on the mean value (step (i)).

4.4 Related Digital Filters

The bilateral is a nonlinear filter on digital images. It has recently been proposed as an alternative to anisotropic diffusion [10]. Unlike the anisotropic diffusion, the bilateral filtering does not involve the solution of partial differential equations and can be implemented in a single iteration [11]. While the bilateral filtering has been originally proposed as an heuristic algorithm, it can be derived as a solution of the regularization of grid graphs. The AWL filter is equivalent to the bilateral filter if in the iteration (8) we have $\mu = 0$ and $w(u, v) = \exp\left(-(u-v)^2/2\sigma_D^2\right) \exp\left(-(g(u)-g(v))^2/2\sigma_R^2\right)$, where g is an image, σ_D is the geometric spread in the domain, and σ_R is the photometric spread in the image range.

The total variation digital filter is another nonlinear filter on digital images, and more generally on arbitrary graphs, which is use for denoising data [4]. It is the discrete version of the total variation formalized by the minimization of (5) with $p = 1$ and $w = 1$ for all edges. Moreover, it can be implemented by the AWL filter by taking the weight function: $w(u, v) = \frac{1}{\|\nabla_v g\|} + \frac{1}{\|\nabla_u g\|}$.

5 Conclusion

We have proposed a family of nonlinear filters, based on weighted graph regularization. This family, which is parameterized by a weight function, includes standard filters as the bilateral filter and the TV digital filter. Moreover, we have shown two applications of the regularization framework in the domain of image processing. As a continuation of this work, we will define a hierarchical segmentation and other weight functions that could realize other fusion processes than the one based on the difference of image intensity. Also, we will apply the regularization to the segmentation of non-organized set of points and to the supervised classification of color images.

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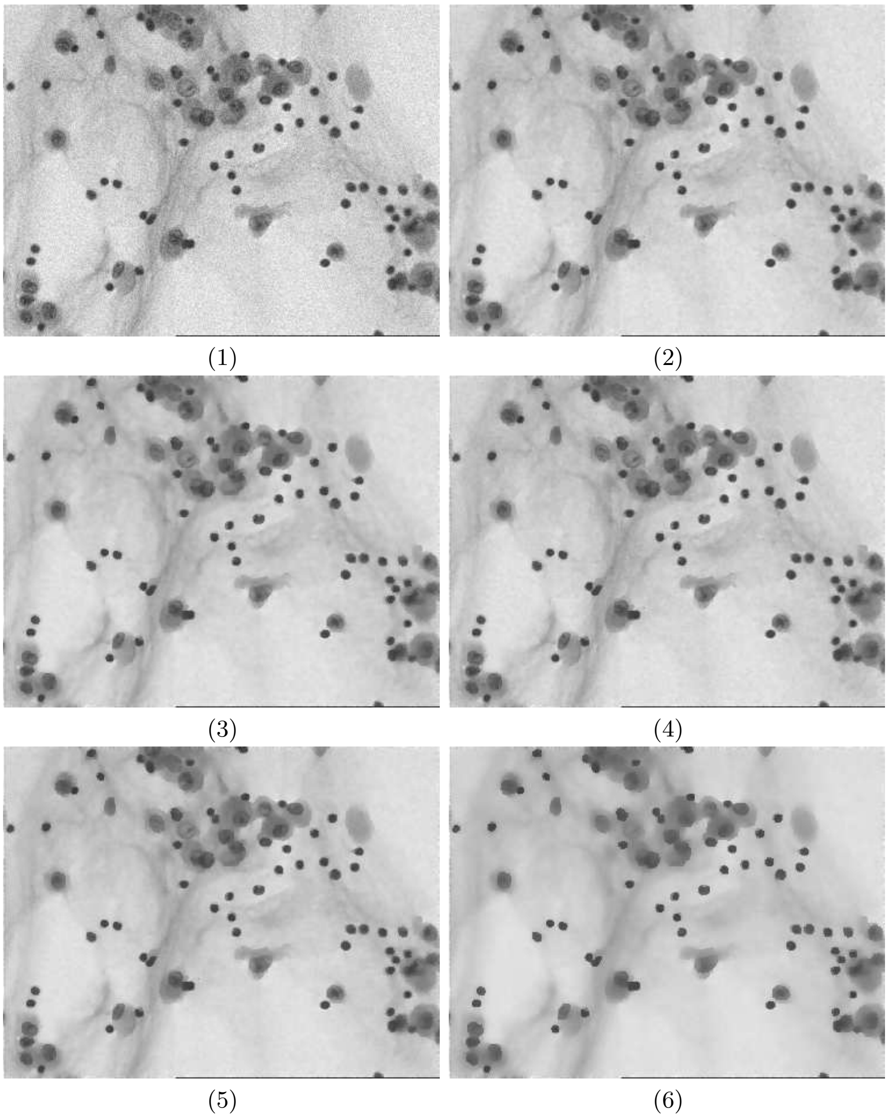

Fig. 1. (1) The original image. The regularizations are all computed with $\lambda = 10$: (2) 5 iterations and $\mu = 0.8$, (3) 5 iterations and $\mu = 0.2$, (4) 100 iterations and $\mu = 0.8$, (5) 100 iterations and $\mu = 0.5$, (6) 100 iterations and $\mu = 0.2$.

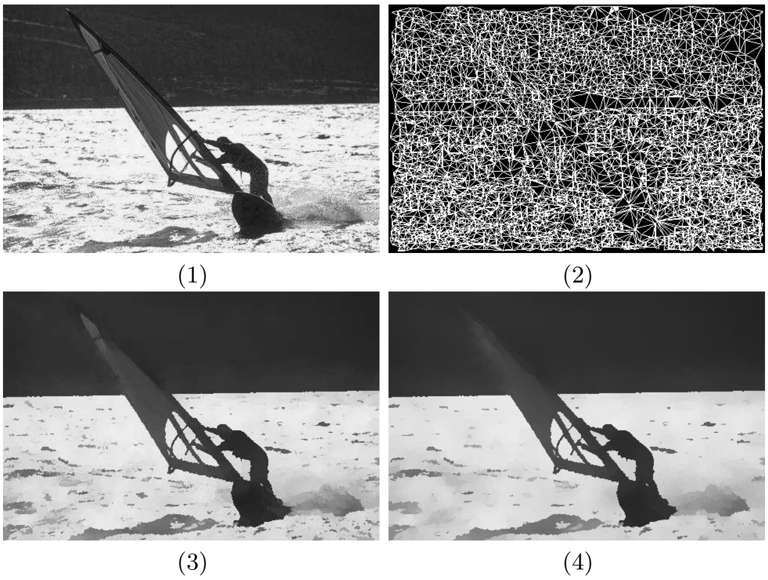

Fig. 2. (1) The original image. (2) The connectivity graph. (3) The regularization with $\mu = 0.5$, $\lambda = 0.5$ and 20 iterations. (4) The regularization with $\mu = 0.5$, $\lambda = 1/5$ and 20 iterations.

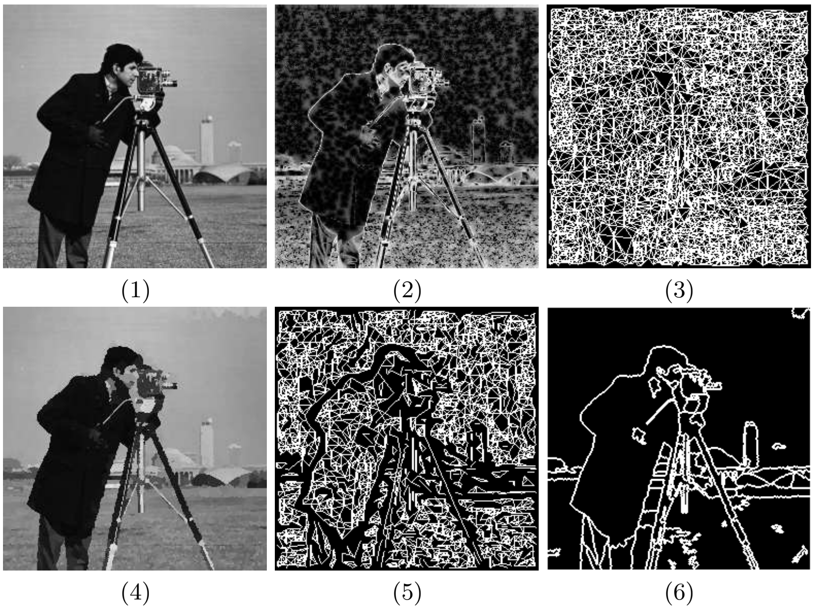

Fig. 3. (1) The original image. (2) The energy image. (3) The initial connectivity graph. (4) The pre-segmented image. (5) The connectivity graph that has been cut after the regularization process with $\lambda = 1/5$, four iterations and $t = 5$. (6) The segmented image.